

## On the bound states for Aharonov-Casher systems

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Bound states for the Aharonov-Casher problem is considered. According to the Hagen's work on the exact equivalence between spin-1/2 Aharonov-Bohm and Aharonov-Casher effects, is known that the  $\nabla \cdot \mathbf{E}$  term can not be neglected in the Hamiltonian if the spin of particle is considered. This term leads to the existence of a singular potential at origin. By modeling the problem by boundary conditions at the origin which arises by the self-adjoint extension of the Hamiltonian, we derive for the first time, an expression for bound state energy of the Aharonov-Casher problem. As an application, we consider the Aharonov-Casher plus a two-dimensional harmonic oscillator. We derive the expression for the harmonic oscillator energies and compare it with the expression obtained in the case without singularity. At the end, an approach for determination of the self-adjoint extension parameter is given. In our approach, the parameter is obtained essentially in terms of physics of the problem.

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## I. INTRODUCTION

Physical processes which exhibit a cyclic evolution play an important role in the description of quantum systems in a periodically changing environment. The system can present a classical description, such as a magnetic dipole precession around an external magnetic field, or quantum behavior such as the electron in a condensate, produced by collective motion of atoms. In this context, the role played by the electromagnetic vector potential is remarkable. The presence of a vector potential in a region where do not produces an electric or magnetic field in the configuration space of free electrons, can influence the interference pattern. Considering a charged particle which propagates in a region with no external magnetic field (force-free region), it is verified that the corresponding wave function may develop a quantum phase:  $\langle b|a\rangle_{inA} = \langle b|a\rangle_{A=0} \left\{ \exp(iq \int_a^b \mathbf{A} \cdot d\mathbf{l}) \right\}$ , which describes the real behavior of the electrons propagation.

Topological effects in quantum mechanics are phenomena that present no classical counterparts, being associated with physical systems defined on a multiply connected space-time<sup>1</sup>. This issue has received considerable attention since the pioneering work by Aharonov and Bohm<sup>2</sup>, where they demonstrated that the vector potential may induce measurable physical quantum phases even in a force-free region, which constitute the essence of a topological effect. The induced phase does not depend on the specific path described by the particle neither on its velocity (non-dispersiveness). Instead, it is intrinsically related to the non-simply connected nature of the space-time and to the associated winding number. This is a first example of generation of a topological phase - the so called Aharonov-Bohm (AB) effect. Many years later, Aharonov and Casher<sup>3</sup> argued that a quantum phase also appears in the wave function of a spin-1/2 neutral particle with anomalous magnetic moment,  $\mu$ , subject to an electric field arising from a charged wire. This is the well-known Aharonov-Casher (AC) effect, which is related to the AB effect by a duality operation<sup>4</sup>. This phase is well established if the wire penetrates perpendicularly to the plane of the neutral particle motion. The AC phase and analogous effects have been studied in several branches of physics in recent years (see for example the Refs.<sup>5-14</sup>).

In order to study the relativistic quantum dynamics of these systems we must consider the full Hamiltonian, i.e., if the particle presents spin, we must consider the electromagnetic field sources. The presence of this term leads to singular solutions at the origin<sup>15-20</sup>. Hagen<sup>4</sup>

showed that there is an exact equivalence between the AB and AC effects when comparing the  $\nabla \cdot \mathbf{E}$  term in the AC Hamiltonian with the  $\nabla \times \mathbf{A}$  term in the AB Hamiltonian. He concluded that the  $\nabla \cdot \mathbf{E}$  term can not be neglected when we consider the spin of the particle<sup>4</sup>. Recently, the  $\nabla \cdot \mathbf{E}$  term was considered by Shikakhwa *et. al*<sup>21</sup> in the scattering scenario. However, an analysis of the energy spectrum considering this term is not found in the literature. This is the main subject of present paper. The AC problem is examined taking into account the  $\nabla \cdot \mathbf{E} = \lambda\delta(r)/2\pi r$  term in the nonrelativistic Hamiltonian. Using the self-adjoint extension method<sup>22</sup>, we model the problem by boundary conditions<sup>23</sup> and determine the expression for the energy spectrum of the particle and compare with the case where one neglects the such term. We also address the AC problem interacting with a two-dimensional harmonic oscillator located at the origin of a polar coordinate system and compare our results with those known from the literature.

The problem of a two-dimensional quantum harmonic oscillator moving under interactions presenting singularities has been addressed in<sup>15</sup>. The self-adjoint extension method was used to guarantee that the Hamiltonian is self-adjoint and to fix the choice of boundary conditions. Specifically, the authors considered a harmonic oscillator added of either a  $\delta$  function potential or a Coulomb potential (which is singular at the origin). In the same work, the authors also applied the results to Landau levels in the presence of a topological defect, the Calogero model and to the quantum motion on the noncommutative plane.

In Ref.<sup>5</sup>, the dynamics of a charged particle in a space endowed with a topological defect (disclination), an Aharonov-Bohm-Casher (ABC) potential, and the Lorentz-violating (LV) nonminimal coupling, and interacting with a two-dimensional harmonic oscillator have been investigated. This system constitutes the ABC problem interacting with a two-dimensional harmonic oscillator in the presence of the nonminimally coupled LV background. It was shown that that in the nonrelativistic limit a LV background is able to modify the harmonic oscillator eigenenergies. In this study, however, the authors did not address the problem by the self-adjoint extension method, because the aim was to analyze the planar dynamics with the origin region excluded.

Park<sup>24</sup> examined the  $\delta$  function potentials in two- and three-dimensional quantum mechanics by incorporating the self-adjoint extension method within the Green's function method. In his work it is considered the spin-1/2 AB problem and a harmonic oscillator potential has been included in the AB Hamiltonian in order to make the energy spectrum

discrete. By considering two limiting cases of self-adjoint extension parameter (tending to zero or infinity), found that the bound-state energies are explicitly determined as poles of the gamma functions. Also, is presented an explicit calculation of the energy-dependent Green's function for the spin-1/2 AB problem.

In AB-like systems, an interesting question which emerge nowadays is the study of the phase generation in real situations, such as in condensed matter, putting, for example, a wire in some material<sup>25</sup>. To reproduce this scenario is advisable to couple the charged wire to a harmonic oscillator (HO), and then analyses this influence on the dynamics of the neutral particle. The vibration of a crystalline lattice, in which the wire is embedded, it is simulated by HO behavior<sup>26,27</sup>.

This paper is organized as follows. In Sec. II the equation of motion for the AC problem is derived in the nonrelativistic limit. In Sec. III the bound states for AC problem is examined. Expressions for the wave functions and bound state energies are obtained fixing the physical problem in the  $r = 0$  region, without any arbitrary parameter. In Sec. IV the AC problem interacting with a two-dimensional harmonic oscillator located at the origin of a polar coordinate system is considered. Again, modeling the problem by boundary conditions, we found the expression for the energy spectrum of the oscillator in terms of the physics of the problem without any arbitrary parameter. In Sec. V we present the procedure to determine the self-adjoint extension parameter. In Sec. VI a brief conclusion is given.

## II. THE EQUATION OF MOTION FOR THE AC PROBLEM

In order to study the dynamics of a neutral particle with a magnetic moment  $\mu$  in a flat space-time we start with the Dirac equation (with  $\hbar = c = 1$ )

$$\left[ i\gamma^\mu \partial_\mu - \frac{\mu}{2} \sigma^{\mu\nu} F_{\mu\nu} - M \right] \psi = 0, \quad \mu, \nu = 0, 1, 2, \quad (1)$$

where  $M$  is the mass of the particle and  $\psi$  is a four-component spinorial wave function and the  $\gamma$ -matrices obey the commutator relation

$$\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]. \quad (2)$$

By introducing a spin projection parameter  $s$  in 2+1 dimensions Eq. (1) reduces to a set of two two-component wave functions. In this case, the Dirac matrices are conveniently defined

in the Pauli representation<sup>4</sup>

$$\beta = \gamma^0 = \sigma_3, \quad \beta\gamma^1 = \sigma_1, \quad \beta\gamma^2 = s\sigma_2. \quad (3)$$

In this representation, the Dirac equation is found to be<sup>4</sup>

$$\left[ M\beta + \beta\boldsymbol{\gamma} \cdot \left( \frac{1}{i}\boldsymbol{\nabla} - \mu s\mathbf{E}' \right) \right] \psi = \bar{\mathcal{E}}\psi, \quad (i = 1, 2) \quad (4)$$

where  $E'_i \equiv \epsilon_{ij}E_j$  denotes the dual field and  $\epsilon_{ij} = -\epsilon_{ji}$ ,  $\epsilon_{12} = +1$ . By applying the matrix operator

$$\left[ M + \beta\bar{\mathcal{E}} - \beta\boldsymbol{\gamma} \cdot \left( \frac{1}{i}\boldsymbol{\nabla} - \mu s\mathbf{E}' \right) \right] \beta \quad (5)$$

in Eq. (4) we obtain

$$(\bar{\mathcal{E}}^2 - M^2) \psi = -(\boldsymbol{\gamma} \cdot \boldsymbol{\pi})(\boldsymbol{\gamma} \cdot \boldsymbol{\pi}) \psi \quad (6)$$

$$= [\boldsymbol{\pi}^2 + \mu\sigma_3(\boldsymbol{\nabla} \cdot \mathbf{E}')] \psi, \quad (7)$$

where  $\boldsymbol{\pi} = \frac{1}{i}\boldsymbol{\nabla} - \mu s\mathbf{E}'$ .

In the usual AC effect the field configuration (in cylindrical coordinates) is given by

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{\hat{\mathbf{r}}}{r}, \quad \boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\lambda}{2\pi\epsilon_0} \frac{\delta(r)}{r}, \quad (8)$$

where  $\mathbf{E}$ , is the electric field generated by an infinite charge filament and  $\lambda$  is the charge density along the  $z$ -axis. To analyze the nonrelativistic limit we assume

$$\begin{aligned} \bar{\mathcal{E}} &= M + \mathcal{E}, \\ \mathcal{E} &\ll M, \end{aligned} \quad (9)$$

and Eq. (6), using Eq. (8), assumes the form

$$\hat{H}_{NR}\psi = \mathcal{E}\psi, \quad (10)$$

where

$$\hat{H}_{NR} = \frac{1}{2M} \left[ \frac{1}{i}\boldsymbol{\nabla} - s\eta \frac{\hat{\mathbf{r}}}{r} \right]^2 + \frac{\eta\sigma_3}{2M} \frac{\delta(r)}{r}, \quad (11)$$

where

$$\eta = \frac{\mu\lambda}{2\pi\epsilon_0}. \quad (12)$$

The nonrelativistic Hamiltonian above describes the planar dynamics of a spin-1/2 neutral particle with a magnetic moment  $\mu$  in an electric field.

Before we going on to a calculation of the bound states some remarks on Hamiltonian in (11) are in order. If we do not take into account the  $\nabla \cdot \mathbf{E}$  term, the resulting Hamiltonian, in this case, is essentially self-adjoint and positive definite<sup>28</sup>. Therefore, its spectrum is  $\mathbb{R}^+$ , it is translationally invariant and there is no bound states. The introduction of  $\nabla \cdot \mathbf{E}$  change the situation completely. The singularity at origin due the  $\nabla \cdot \mathbf{E}$ , is physically equivalent to extract this single point from the plane  $\mathbb{R}^2$  and in this case the translational invariance is lost together with the self-adjointness. This fact has impressive consequences in the spectrum of the system<sup>29</sup>. Since we are effectively excluding a portion of space accessible to the particle we must guarantee that the Hamiltonian is self-adjoint in the region of the motion, as is necessary for the generator of time evolution of the wave function. The most adequate approach for studying this scenario is the theory of self-adjoint extension of simetrical operators of von Neumann-Krein<sup>22,30,31</sup>. The existence of a negative eigenvalue in the spectrum can be considered rather unexpected, since the actions of suggest it is positive definite operator. However, the positivity of such an operator just not depends of its action, but also depends on its domain. Moreover, the  $\nabla \cdot \mathbf{E}$  gives rising to a two-dimensional  $\delta$  function potential at origin, and it is a well known fact that an attractive  $\delta$  function allows at least one bound state<sup>30,32</sup>.

In particular, we analyze the changes in the energy levels and wave functions of the particle in the region near the charge filament. For this system, the commutator

$$\left[ \hat{H}_{NR}, \hat{J}_z \right] = 0, \quad (13)$$

where  $\hat{J}_z = -i\partial_\varphi + \sigma_z/2$  is the total angular momentum operator in the  $z$  direction. So, the solution for the Schrödinger equation (10) can be written in the form

$$\Phi(r, \varphi) = \begin{bmatrix} f_{\mathcal{E}}(r)e^{i(m_j-1/2)\varphi} \\ g_{\mathcal{E}}(r)e^{i(m_j+1/2)\varphi} \end{bmatrix}, \quad (14)$$

with  $m_j = m + 1/2 = \pm 1/2, \pm 3/2, \dots, m \in \mathbb{Z}$ . By substitution Eq. (14) into the Eq. (10), the radial equation for  $f_{\mathcal{E}}(r)$  becomes

$$\mathcal{H}f_{\mathcal{E}}(r) = \mathcal{E}f_{\mathcal{E}}(r), \quad (15)$$

where

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{U}_{\text{short}}, \quad (16)$$

$$\mathcal{H}_0 = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\xi^2}{r^2} \right], \quad (17)$$

$$\mathcal{U}_{\text{short}} = \frac{\eta}{2M} \frac{\delta(r)}{r}, \quad (18)$$

with

$$\xi = m + s\eta. \quad (19)$$

In what follows we assume that  $\eta < 0$  to ensure we have an attractive  $\delta$  function with at last one bound state.

Since we are considering an infinite hollow cylinder of radius  $r_0$  with charge density per unit length  $\lambda$ , is suitable to rewrite the short-range potential as (see<sup>33</sup> and references there in)

$$\overline{\mathcal{U}}_{\text{short}}(r) = \frac{\eta}{2M} \frac{\delta(r - r_0)}{r_0}, \quad (20)$$

and, at the end, the limit  $r_0 \rightarrow 0$  is taken. Although the functional structure of  $\mathcal{U}_{\text{short}}$  and  $\overline{\mathcal{U}}_{\text{short}}$  are quite different, as discussed in<sup>34</sup>, we are free to use any form of potential provided that only the contribution of the form (18) is excluded.

### III. BOUND STATES FOR AC PROBLEM

Now, the goal is to find the bound states for the Hamiltonian (16). This Hamiltonian includes a short-range interaction  $\overline{\mathcal{U}}_{\text{short}}$  modeled by a  $\delta$  function. Such kind of short-range interaction also appears in several Aharonov-Bohm-like problems, e.g., Aharonov-Bohm scattering of spin-1/2 particles<sup>18,33,35</sup>, the coupling between wave functions and conical defects/cosmic strings<sup>36</sup> and coupling between wave functions and torsion<sup>7</sup>.

To dealt with the singularity point at  $r = 0$ , we follow the approach in<sup>23</sup>. Then we temporarily forget the  $\delta$  function potential and find which boundary conditions are allowed for  $\mathcal{H}_0$ . This is the scope of the self-adjoint extension, which consists in determining the complete domain of an operator, i.e., its complete set of wave functions. But the self-adjoint extension provides us with an infinity of possible boundary conditions and therefore it can not give us the true physics of the problem. Nevertheless, once fixed the physics at  $r = 0$ <sup>37,38</sup>, we are able to fit any arbitrary parameter coming from the self-adjoint extension and then we have a complete description of the problem.

Since we have a singular point, even if  $\mathcal{H}_0^\dagger = \mathcal{H}_0$ , we must guarantee that the Hamiltonian is self-adjoint in the region of motion for their domains might be different. The von

Neumann-Krein method<sup>22</sup> is used to find the self-adjoint extensions. An operator  $\mathcal{H}_0$  with domain  $\mathcal{D}(\mathcal{H}_0)$  is self-adjoint if  $\mathcal{D}(\mathcal{H}_0^\dagger) = \mathcal{D}(\mathcal{H}_0)$  and  $\mathcal{H}_0^\dagger = \mathcal{H}_0$ . In order to proceed with the self-adjoint extension, we must find the deficiency subspaces  $\mathcal{N}_\pm$ , with dimensions  $n_+$  and  $n_-$ , which are called deficiency indices of  $\mathcal{H}_0$ . A necessary and sufficient condition for  $\mathcal{H}_0$  to be self-adjoint is that  $n_+ = n_- = 0$ . On the other hand, if  $n_+ = n_- \geq 1$  then  $\mathcal{H}_0$  has an infinite number of self-adjoint extensions parametrized by a unitary  $n \times n$  matrix, where  $n = n_+ = n_-$ .

The potential in this case is purely radial and we decompose the Hilbert space  $H = L^2(\mathbb{R}^2)$  with respect to the angular momentum  $H = H_r \otimes H_\varphi$ , where  $H_r = L^2(\mathbb{R}^+, r dr)$  and  $H_\varphi = L^2(\mathcal{S}^1, d\varphi)$ , with  $\mathcal{S}^1$  denoting the unit sphere in  $\mathbb{R}^2$ . The operator  $-\frac{\partial^2}{\partial \varphi^2}$  is essentially self-adjoint in  $L^2(\mathcal{S}^1, d\varphi)$ <sup>22</sup> and we obtain the operator  $\mathcal{H}_0$  in each angular momentum sector. We have to pay special attention for the radial eigenfunctions, due to the singularity at  $r = 0$ .

Next, we substitute the problem in Eq. (15) by

$$\mathcal{H}_0 f_{\varrho, \varepsilon} = \varepsilon f_{\varrho, \varepsilon}, \quad (21)$$

with  $f_{\varrho, \varepsilon}$  labeled by a parameter  $\varrho$  which is related to the behavior of the wave function in the limit  $r \rightarrow r_0$ . But we can not impose any boundary condition (e.g.  $f = 0$  at  $r = 0$ ) without discovering which boundary conditions are allowed to  $\mathcal{H}_0$ .

In order to find the full domain of  $\mathcal{H}_0$  in  $L^2(\mathbb{R}^+, r dr)$ , we have to find its deficiency subspaces. To do this, we solve the eigenvalue equation

$$\mathcal{H}_0^\dagger f_\pm = \pm i k_0 f_\pm, \quad (22)$$

where  $\mathcal{H}_0^\dagger$  is given by Eq. (17) and  $k_0 \in \mathbb{R}$  is introduced for dimensional reasons. The only square-integrable functions which are solutions of Eq. (22) are the modified Bessel functions

$$f_\pm(r) = \text{const. } K_\xi(r\sqrt{\mp\varepsilon}), \quad (23)$$

with  $\varepsilon = 2iMk_0$ . These functions are square-integrable only in the range  $\xi \in (-1, 1)$ , for which  $\mathcal{H}_0$  is not self-adjoint, and the dimension of such deficiency subspaces is  $(n_+, n_-) = (1, 1)$ . So we have two situations for  $\xi$ , i.e.,

$$-1 < \xi < 0, \quad (24)$$

$$0 < \xi < 1.$$



To treat both cases of Eq. (24) simultaneously, it is more convenient to use

$$f_{\pm}(r) = \text{const. } K_{|\xi|}(r\sqrt{\mp\varepsilon}). \quad (25)$$

Thus, the domain  $\mathcal{D}(\mathcal{H}_{0,\varrho})$  in  $L^2(\mathbb{R}^+, r dr)$  is given by the set of functions<sup>22</sup>

$$f_{\varrho,\varepsilon}(r) = f_{|\xi|}(r) + C [K_{|\xi|}(r\sqrt{-\varepsilon}) + e^{i\vartheta} K_{|\xi|}(r\sqrt{\varepsilon})], \quad (26)$$

where  $f_{|\xi|}(r)$ , with  $f_{|\xi|}(r_0) = \dot{f}_{|\xi|}(r_0) = 0$  ( $\dot{f} \equiv df/dr$ ), is the regular wave function when we do not have  $\overline{\mathcal{U}}_{\text{short}}(r)$ . The last term in Eq. (26) gives the correct behavior for the wave function when  $r = r_0$ . The parameter  $\varrho \in [0, 2\pi)$  represents a choice for the boundary condition in the region around  $r = r_0$  and describes the coupling between  $\overline{\mathcal{U}}_{\text{short}}(r)$  and the wave function. As we shall see below, the physics of the problem determines such parameter without ambiguity.

To find a fitting for  $\varrho$  compatible with  $\overline{\mathcal{U}}_{\text{short}}(r)$ , we write Eqs. (15) and (21) for  $\mathcal{E} = 0$ <sup>23</sup>,

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\xi^2}{r^2} + \overline{\mathcal{U}}_{\text{short}} \right] f_0 = 0, \quad (27)$$

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\xi^2}{r^2} \right] f_{\varrho,0} = 0, \quad (28)$$

implying the zero-energy solutions  $f_0$  and  $f_{\varrho,0}$ , and we require the continuity for the logarithmic derivative<sup>23</sup>

$$\frac{r_0}{f_0(r)} \frac{df_0(r)}{dr} \Big|_{r=r_0} = \frac{r_0}{f_{\varrho,0}(r)} \frac{df_{\varrho,0}(r)}{dr} \Big|_{r=r_0}. \quad (29)$$

The left-hand side of Eq. (29) is achieved integrating the Eq. (27) from 0 to  $r_0$ ,

$$\int_0^{r_0} \frac{1}{r} \frac{d}{dr} \left( r \frac{df_0(r)}{dr} \right) r dr = \eta \int_0^{r_0} f_0(r) \frac{\delta(r-r_0)}{r} r dr + \xi^2 \int_0^{r_0} \frac{f_0(r)}{r^2} r dr. \quad (30)$$

From (27), the behavior of  $f_0$  as  $r \rightarrow 0$  is  $f_0 \sim r^{|\xi|}$ , so we find

$$\int_0^{r_0} \frac{f_0(r)}{r^2} r dr \approx \int_0^{r_0} r^{|\xi|-1} dr \rightarrow 0. \quad (31)$$

So, we arrive at

$$r_0 \frac{\dot{f}_0}{f_0} \Big|_{r=r_0} \approx \eta. \quad (32)$$

The right-hand side of Eq. (29) is calculated using the asymptotic representation for the modified Bessel functions in the limit  $z \rightarrow 0$ ,

$$K_\nu(z) \sim \frac{\pi}{2 \sin(\pi\nu)} \left[ \frac{z^{-\nu}}{2^{-\nu} \Gamma(1-\nu)} - \frac{z^\nu}{2^\nu \Gamma(1+\nu)} \right], \quad (33)$$

in Eq. (26) and it takes the form

$$\frac{r_0}{f_{\varrho,0}(r_0)} \frac{df_{\varrho,0}(r)}{dr} \Big|_{r=r_0} = \frac{1}{\Omega_{\varrho}(r_0)} \frac{d\Omega_{\varrho}(r)}{dr} \Big|_{r=r_0}, \quad (34)$$

where

$$\begin{aligned} \Omega_{\varrho}(r) = & \left[ \frac{(r\sqrt{-\varepsilon})^{-|\xi|}}{2^{-|\xi|}\Gamma(1-|\xi|)} - \frac{(r\sqrt{-\varepsilon})^{|\xi|}}{2^{|\xi|}\Gamma(1+|\xi|)} \right] \\ & + e^{i\varrho} \left[ \frac{(r\sqrt{\varepsilon})^{-|\xi|}}{2^{-|\xi|}\Gamma(1-|\xi|)} - \frac{(r\sqrt{\varepsilon})^{|\xi|}}{2^{|\xi|}\Gamma(1+|\xi|)} \right], \end{aligned} \quad (35)$$

Substituting (32) and (34) in (29) we have

$$\frac{1}{\Omega_{\varrho}(r_0)} \frac{d\Omega_{\varrho}(r)}{dr} \Big|_{r=r_0} = \eta, \quad (36)$$

which determines the parameter  $\varrho$  in terms of the physics of the problem, i.e., the correct behavior of the wave functions for  $r \rightarrow r_0$ .

Next, we will find the bound states of the Hamiltonian  $\mathcal{H}_0$  and using (36), the spectrum of  $\mathcal{H}$  will be determined without any arbitrary parameter. Then, from Eq. (21) we achieve the modified Bessel equation ( $\kappa^2 = -2M\mathcal{E}$ ,  $\mathcal{E} < 0$ )

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( \frac{\xi^2}{r^2} + \kappa^2 \right) \right] f_{\varrho,\mathcal{E}}(r) = 0, \quad (37)$$

whose general solution is given by

$$f_{\varrho,\mathcal{E}}(r) = K_{|\xi|} \left( r\sqrt{-2M\mathcal{E}} \right). \quad (38)$$

Since this solutions belongs to  $\mathcal{D}(\mathcal{H}_{0,\varrho})$ , it is the form (26), that is,

$$f_{\varrho,\mathcal{E}}(r) = f_{|\xi|}(r) + C \left[ K_{|\xi|}(r\sqrt{-\varepsilon}) + e^{i\varrho} K_{|\xi|}(r\sqrt{\varepsilon}) \right], \quad (39)$$

for some  $\varrho$  selected from the physics of the problem at  $r = r_0$ . So, we substitute (38) in (39) and compute  $\frac{r_0}{f_{\varrho,\mathcal{E}}(r_0)} \frac{df_{\varrho,\mathcal{E}}(r_0)}{dr}$  using (33). After a straightforward calculation we have the relation

$$\begin{aligned} \frac{r_0}{f_{\varrho,\mathcal{E}}(r_0)} \frac{df_{\varrho,\mathcal{E}}(r_0)}{dr} &= \frac{|\xi| \left[ r_0^{2|\xi|} \Gamma(1-|\xi|) (-M\mathcal{E})^{|\xi|} + 2^{|\xi|} \Gamma(1+|\xi|) \right]}{r_0^{2|\xi|} \Gamma(1-|\xi|) (-M\mathcal{E})^{|\xi|} - 2^{|\xi|} \Gamma(1+|\xi|)} \\ &= \frac{1}{\Omega_{\varrho}(r_0)} \frac{d\Omega_{\varrho}(r_0)}{dr}. \end{aligned} \quad (40)$$

By substituting the Eqs. (34) and (32) in Eq. (29) and solving for  $\mathcal{E}$ , we find the sought energy spectrum

$$\mathcal{E} = -\frac{2}{Mr_0^2} \left[ \left( \frac{\eta + |\xi|}{\eta - |\xi|} \right) \frac{\Gamma(1 + |\xi|)}{\Gamma(1 - |\xi|)} \right]^{1/|\xi|}. \quad (41)$$

Notice that there is no arbitrary parameters in the above equation and we must have  $\eta \leq -1$  to ensure that the energy is a real number. The AC problem only have bound states when we take into account the  $\nabla \cdot \mathbf{E}$ , which gives raising of an attractive  $\delta$  function.

#### IV. AC PLUS A TWO-DIMENSIONAL HARMONIC OSCILLATOR

In this section, we address the AC system plus a two-dimensional harmonic oscillator (HO) located at the origin of a polar coordinate system by using the same approach as in Sec. III.

The potential of the harmonic oscillator in two-dimensional space is given by

$$\hat{V}_{HO} = \frac{1}{2}M\omega_x^2 x^2 + \frac{1}{2}M\omega_y^2 y^2. \quad (42)$$

In polar coordinates  $(r, \varphi)$  it can be written as

$$\hat{V}_{HO} = \frac{1}{2}M\omega^2 r^2, \quad (43)$$

where we considered  $\omega_x = \omega_y = \omega$ .

Let us now include the potential of the oscillator (43) into the AC Hamiltonian (11), which leads to the following eigenvalue equation

$$\hat{H}_{HO}\Phi = i\partial_t\Phi, \quad (44)$$

where

$$\hat{H}_{HO} = \hat{H}_{NR} + \hat{V}_{HO}. \quad (45)$$

By proceeding in the same manner as in Sec. III, we can write the solutions as

$$\Phi(r, \varphi) = \begin{bmatrix} \phi_{\mathcal{E}'}(r)e^{i(m_j-1/2)\varphi} \\ \zeta_{\mathcal{E}'}(r)e^{i(m_j+1/2)\varphi} \end{bmatrix}, \quad (46)$$

which implies the following radial equation for eigenvalues

$$\mathcal{H}'\phi_{\mathcal{E}'}(r) = \mathcal{E}'\phi_{\mathcal{E}'}(r), \quad (47)$$

where

$$\mathcal{H}' = \mathcal{H}'_0 + \overline{\mathcal{U}}_{\text{short}}, \quad (48)$$

$$\mathcal{H}'_0 = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\xi^2}{r^2} - \gamma^2 r^2 \right], \quad (49)$$

and  $\gamma^2 = M^2 \omega^2$ .

In order to have a more detailed study of the problem we will analyze separately the motion of the particle in the  $r \neq 0$  region and including the  $r = 0$  region. This approach allows us to see explicitly the physical implications of the  $\nabla \cdot \mathbf{E}$  term on the energy spectrum of the particle and thus makes clear the fact can not be neglected in the Hamiltonian.

### A. Solution of the problem in the $r \neq 0$ region

In this case, the Eq. (47) does not contain the  $\overline{\mathcal{U}}_{\text{short}}$  term. Then, the general solution for  $\phi_{\mathcal{E}'}(r)$  in the  $r \neq 0$  region is<sup>39</sup>

$$\begin{aligned} \phi_{\mathcal{E}'}(r) = & A_\xi \gamma^{\frac{1+\xi}{2}} r^\xi e^{-\frac{1}{2}\gamma r^2} M(d, 1 + \xi, \gamma r^2) \\ & + B_\xi \gamma^{\frac{1+\xi}{2}} r^\xi e^{-\frac{1}{2}\gamma r^2} U(d, 1 + \xi, \gamma r^2), \end{aligned} \quad (50)$$

where  $d = \frac{1+\xi}{2} - \frac{M\xi'}{2\gamma}$ ,  $M(a, b, z)$  and  $U(a, b, z)$  are the confluent hypergeometric functions (Kummer's functions)<sup>40</sup>,  $A_\xi$  and  $B_\xi$  constants. However, only  $M$  is regular at origin, this implies that  $B_\xi = 0$ . In the above solution, moreover, if  $d$  is 0 or a negative integer, the series terminates and the hypergeometric function becomes a polynomial of degree  $n$ <sup>40</sup>. This condition guarantees that the hypergeometric function is regular at origin, which is essential for the treatment of the physical system since the region of interest is that around the charge filament. Therefore, the series in (50) must converge if we consider that  $d = -n$ , where  $n \in \mathbb{Z}^*$ ,  $\mathbb{Z}^*$  denoting the set of nonnegative integers,  $n = 0, 1, 2, 3, \dots$ . This condition also guarantees the normalizability of the wave function. So, using this condition, we obtain the discrete values for the energy whose expression is given by

$$\mathcal{E}' = (2n + 1 + |\xi|)\omega, \quad n \in \mathbb{Z}^*. \quad (51)$$

The energy eigenfunction is given by

$$\Phi(r, \varphi) = C_\xi \gamma^{\frac{1+|\xi|}{2}} r^{|\xi|} e^{-\frac{1}{2}\gamma r^2} M(-n, 1 + |\xi|, \gamma r^2) e^{i(m-s/2)\varphi}, \quad (52)$$

where  $C_\xi$  is a normalization constant. Notice that in Eq. (51),  $|\xi|$  can assume any noninteger number. However, we will see that this condition is no longer satisfied when we include the  $\nabla \cdot \mathbf{E}$  term. To study the dynamics of the particle in all space, including the  $r = 0$  region, we invoke the self-adjoint extension of symmetric operators. The procedure used here to derive the result of the Eq. (51) is found in many articles in the literature, where the authors simply ignore the term involving the singularity. As we will show in the next section, this procedure reflect directly in the energy spectrum of the system.

## B. Solution including the $r = 0$ region

Now, the dynamics includes the  $\overline{\mathcal{U}}_{\text{short}}$  term. So, let us follow the same procedure as in Sec. III to find the bound states for the Hamiltonian  $\mathcal{H}'$ . Like before we need to find all the self-adjoint extension for the operator  $\mathcal{H}'_0$ . The relevant eigenvalue equation is

$$\mathcal{H}'_0 \phi_{\vartheta, \mathcal{E}'}(r) = \mathcal{E}' \phi_{\vartheta, \mathcal{E}'}(r), \quad (53)$$

with  $\phi_{\vartheta, \mathcal{E}'}$  labeled by a parameter  $\vartheta$  which is related to the behavior of the wave function in the limit  $r \rightarrow r_0$ . The solutions of this equation are given in (50). However, the only square integrable functions is  $U(d, 1 + \xi, \gamma r^2)$ . Then, this implies that  $A_\xi = 0$  in Eq. (50), so that

$$\phi_{\vartheta, \mathcal{E}'}(r) = \gamma^{\frac{1+\xi}{2}} r^\xi e^{-\frac{1}{2}\gamma r^2} U(d, 1 + \xi, \gamma r^2). \quad (54)$$

To guarantee that  $\phi_{\vartheta, \mathcal{E}'}(r) \in L^2(\mathbb{R}, r dr)$  it is advisable to study their behavior as  $r \rightarrow 0$ , which implies analyzing the possible self-adjoint extensions.

Now, in order to construct the self-adjoint extensions, let us consider the eigenvalue equation

$$\mathcal{H}'_0 \phi_\pm(r) = \pm i k_0 \phi_\pm(r). \quad (55)$$

Since  $\mathcal{H}'_0^\dagger = \mathcal{H}'_0$  and, from (54), the square integrable solution to above equation is given by

$$\phi_\pm(r) = r^\xi e^{-\frac{\gamma r^2}{2}} U(d_\pm, 1 + \xi, \gamma r^2), \quad (56)$$

where  $d_\pm = \frac{1+\xi}{2} \mp \frac{i k_0}{2\gamma}$ . Let us now consider the asymptotic behavior of  $U(d_\pm, 1 + \xi, \gamma r^2)$  as  $r \rightarrow 0$ <sup>39</sup>,

$$U(d_\pm, 1 + \xi, \gamma r^2) \sim \left[ \frac{\Gamma(\xi)(\gamma r)^{-\xi}}{\Gamma(d_\pm)} + \frac{\Gamma(-\xi)r^\xi}{\Gamma(d_\pm - \xi)} \right]. \quad (57)$$

Working with the expression, let us find under which condition

$$\int |\phi_{\pm}(r)|^2 r dr, \quad (58)$$

has a finite contribution from the near origin region. Taking (56) and (57) into account, we have

$$\lim_{r \rightarrow 0} |\phi_{\pm}(r)|^2 r^{1+2\xi} \longrightarrow [\mathcal{A}_1 r^{1+2\xi} + \mathcal{A}_2 r^{1-2\xi}], \quad (59)$$

where  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are constants. The equation above show that  $\phi_{\pm}(r)$  is square integrable only for  $\xi \in (-1, 1)$ . In this case, since  $\mathcal{N}_+$  is expanded by  $\phi_+(r)$  only, we have that its dimension  $n_+ = 1$ . The same applies to  $\mathcal{N}_-$  and  $\phi_-(r)$  resulting in  $n_- = 1$ . Then,  $\mathcal{H}'_0$  possesses self-adjoint extensions parametrized by a unitary matrix  $U(1) = e^{i\vartheta}$ , with  $\vartheta \in [0, 2\pi)$ . Therefore, the domain  $\mathcal{D}(\mathcal{H}'_0{}^\dagger)$  in  $L^2(\mathbb{R}^+, r dr)$  is given by

$$\mathcal{D}(\mathcal{H}'_0{}^\dagger) = \mathcal{D}(\mathcal{H}'_0) \oplus \mathcal{N}_+ \oplus \mathcal{N}_-. \quad (60)$$

So, to extend the domain  $\mathcal{D}(\mathcal{H}'_0)$  to match  $\mathcal{D}(\mathcal{H}'_0{}^\dagger)$  and therefore make  $\mathcal{H}'_0$  self-adjoint, we get

$$\mathcal{D}(\mathcal{H}'_{0,\vartheta}) = \mathcal{D}(\mathcal{H}'_0{}^\dagger) = \mathcal{D}(\mathcal{H}'_0) \oplus \mathcal{N}_+ \oplus \mathcal{N}_-, \quad (61)$$

we mean that, for each  $\vartheta$ , we have a possible domain for  $\mathcal{D}(\mathcal{H}'_{0,\vartheta})$ . But will be the physical situation which will determine the value of  $\vartheta$ . The Hilbert space (61), for both cases of Eq. (24), contains functions of the form

$$\phi_{\vartheta,\varepsilon'}(r) = \phi_{|\xi|}(r) + C r^{|\xi|} e^{-\frac{\gamma r^2}{2}} \left[ U(d_+, 1 + |\xi|, \gamma r^2) + e^{i\vartheta} U(d_-, 1 + |\xi|, \gamma r^2) \right], \quad (62)$$

where  $C$  is an arbitrary complex number,  $\phi_{|\xi|}(0) = \dot{\phi}_{|\xi|}(0) = 0$  with  $\phi_{|\xi|}(r) \in L^2(\mathbb{R}^+, r dr)$ , and  $d_{\pm} = \frac{1+|\xi|}{2} \mp \frac{ik_0}{2\gamma}$ . For a range of  $\vartheta$ , the behavior of the wave functions (62) was addressed in<sup>41</sup>. However, as we will see below, if we fix the physics of the problem at  $r = r_0$ , there is no need for such analysis because the value of  $\vartheta$  is automatically selected.

In order to find the spectrum of  $\mathcal{H}'_0$  we consider the limit  $r \rightarrow 0$  of  $\phi_{\vartheta,\varepsilon'}(r)$ , that is, we substitute (54) in the left side of (62) and, using (57), leads to

$$A \left[ \frac{\Gamma(|\xi|)(\gamma r)^{-|\xi|}}{\Gamma(d)} + \frac{\Gamma(-|\xi|)r^{|\xi|}}{\Gamma(d-|\xi|)} \right] = C \left\{ \left[ \frac{\Gamma(|\xi|)(\gamma r)^{-|\xi|}}{\Gamma(d_+)} + \frac{\Gamma(-|\xi|)r^{|\xi|}}{\Gamma(d_+-|\xi|)} \right] + e^{i\vartheta} \left[ \frac{\Gamma(|\xi|)(\gamma r)^{-|\xi|}}{\Gamma(d_-)} + \frac{\Gamma(-|\xi|)r^{|\xi|}}{\Gamma(d_- - |\xi|)} \right] \right\}. \quad (63)$$

Now, equating the coefficients of the same powers of  $r$ , we get

$$\frac{A}{\Gamma(d)} = C \left[ \frac{1}{\Gamma(d_+)} + \frac{e^{i\vartheta}}{\Gamma(d_-)} \right], \quad (64)$$

and

$$\frac{A}{\Gamma(d - |\xi|)} = C \left[ \frac{1}{\Gamma(d_+ - |\xi|)} + \frac{e^{i\vartheta}}{\Gamma(d_- - |\xi|)} \right], \quad (65)$$

whose quotient leads to

$$\frac{\Gamma(d - |\xi|)}{\Gamma(d)} = \frac{\frac{1}{\Gamma(d_+)} + \frac{e^{i\vartheta}}{\Gamma(d_-)}}{\frac{1}{\Gamma(d_+ - |\xi|)} + \frac{e^{i\vartheta}}{\Gamma(d_- - |\xi|)}}. \quad (66)$$

The left-hand side of this equation is a function of the energy  $\mathcal{E}'$  while its right-hand side is a constant (even though it depends on the extension parameter  $\vartheta$  which is fixed by the physics of the problem). Then we have the equation

$$\frac{\Gamma\left(\frac{1-|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1+|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)} = \text{const.} \quad (67)$$

Therefore we have achieved the energy levels with an arbitrary parameter  $\vartheta$ , and we get inequivalent quantizations to different values of it<sup>42</sup>. Each physical problem selects a specific  $\vartheta$ .

Next, we replace problem  $\mathcal{H}' = \mathcal{H}'_0 + \overline{\mathcal{U}}_{\text{short}}$ , by  $\mathcal{H}'_0$  plus self-adjoint extensions. So, consider the static solutions ( $\mathcal{E}' = 0$ )  $\phi_0(r)$  and  $\phi_{\vartheta,0}(r)$  for the equations

$$[\mathcal{H}'_0 + \overline{\mathcal{U}}_{\text{short}}] \phi_0(r) = 0, \quad (68)$$

$$\mathcal{H}'_0 \phi_{\vartheta,0}(r) = 0. \quad (69)$$

Now, to find out the value of  $\vartheta$ , it is required the continuity of logarithmic derivative

$$\frac{r_0}{\phi_0(r_0)} \frac{d\phi_0(r)}{dr} \Big|_{r=r_0} = \frac{r_0}{\phi_{\vartheta,0}(r_0)} \frac{d\phi_{\vartheta,0}(r)}{dr} \Big|_{r=r_0}. \quad (70)$$

The left-hand side of Eq. (70) is obtained by integration of Eq. (68) from 0 to  $r_0$ . Since the integration of the harmonic term

$$\int_0^{r_0} \gamma^2 r^2 r^{1+2|\xi|} \phi_0(r) dr \approx \phi_0(r=r_0) \int_0^{r_0} \gamma^2 r^2 r^{1+2|\xi|} dr \rightarrow 0, \quad (71)$$

as  $r_0 \rightarrow 0$ , the result is the same as in (32). Then, we arrive at

$$\frac{r_0}{\phi_0(r_0)} \frac{d\phi_0(r)}{dr} \Big|_{r=r_0} = \eta. \quad (72)$$

The right-hand side of Eq. (70) is calculated using the asymptotic behavior of  $\phi_{\pm}(r)$  (see Eq. (57)) into Eq. (62), and it take the form

$$\frac{r_0}{\phi_{\vartheta,0}(r_0)} \frac{d\phi_{\vartheta,0}(r)}{dr} \Big|_{r=r_0} = \frac{1}{\Omega'_{\vartheta}(r_0)} \frac{d\Omega'_{\vartheta}(r)}{dr} \Big|_{r=r_0}, \quad (73)$$

where

$$\Omega'_{\vartheta}(r) = \left[ \frac{\Gamma(|\xi|)(\gamma r)^{-|\xi|}}{\Gamma(d_+)} + \frac{\Gamma(-|\xi|)r^{|\xi|}}{\Gamma(d_+ - \xi)} \right] + e^{i\vartheta} \left[ \frac{\Gamma(|\xi|)(\gamma r)^{-|\xi|}}{\Gamma(d_-)} + \frac{\Gamma(-|\xi|)r^{|\xi|}}{\Gamma(d_- - |\xi|)} \right], \quad (74)$$

Replacing (72) and (73) in (70) we arrive at

$$\frac{r_0}{\Omega'_{\vartheta}(r_0)} \frac{d\Omega'_{\vartheta}(r)}{dr} \Big|_{r=r_0} = \eta. \quad (75)$$

With this relation we select an approximated value for parameter  $\vartheta$  in terms of the physics of the problem. The solutions for (53) are given by (54) and since this function belongs to  $\mathcal{D}(\mathcal{H}'_{0,\vartheta})$ , it is of the form (62), that is,

$$\phi_{\vartheta,\mathcal{E}'}(r) = \phi_{|\xi|}(r) + Cr^{|\xi|} e^{-\frac{\gamma r^2}{2}} \left[ U(d_+, 1 + |\xi|, \gamma r^2) + e^{i\vartheta} U(d_-, 1 + |\xi|, \gamma r^2) \right], \quad (76)$$

for some  $\vartheta$  selected from the physics of the problem at  $r = r_0$ . So, using (54) in (76) we have

$$\begin{aligned} \frac{r_0}{\phi_{\vartheta,\mathcal{E}'}(r_0)} \frac{d\phi_{\vartheta,\mathcal{E}'}(r)}{dr} \Big|_{r=r_0} &= \frac{|\xi| [r_0^{2|\xi|} \gamma^{|\xi|} \Gamma(d) \Gamma(1 - |\xi|) + \Gamma(1 + |\xi|) \Gamma(d - |\xi|)]}{r_0^{2|\xi|} \gamma^{|\xi|} \Gamma(d) \Gamma(1 - |\xi|) - \Gamma(1 + |\xi|) \Gamma(d - |\xi|)} \\ &= \frac{1}{\Omega'_{\vartheta}(r_0)} \frac{d\Omega'_{\vartheta}(r)}{dr} \Big|_{r=r_0}. \end{aligned} \quad (77)$$

Using the Eq. (75) into the above equation we obtain

$$\frac{\Gamma(\frac{1+|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma})}{\Gamma(\frac{1-|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma})} = -\frac{1}{\gamma^{|\xi|} r_0^{2|\xi|}} \left( \frac{\eta + |\xi|}{\eta - |\xi|} \right) \frac{\Gamma(1 + |\xi|)}{\Gamma(1 - |\xi|)}. \quad (78)$$

The Eq. (78) is too complicated to evaluate the bound state energy explicitly, but its limiting features are interesting. If we take limit  $r_0 \rightarrow 0$  in this expression, the bound state energy are determined by the poles of the gamma functions, i.e.,

$$\begin{aligned} -1 < \xi < 0, \quad \mathcal{E}' &= (2n + 1 - |\xi|)\omega, \\ 0 < \xi < 1, \quad \mathcal{E}' &= (2n + 1 + |\xi|)\omega, \end{aligned} \quad (79)$$

or

$$\mathcal{E}' = (2n + 1 \pm |\xi|)\omega, \quad (80)$$



where  $n \in \mathbb{Z}^*$ . The  $+$  ( $-$ ) sign refers to solutions which are regular (singular) at the origin. This result coincides with the study done by Blum *et al.*<sup>43</sup>. It should be noted that in the absence of spin (i.e., when  $\nabla \cdot \mathbf{E}$  is absence), like showed in Sec. IV A, we always have a regular solution it is the plus sign which must be used in (80). In the next section we will confirm this using another approach.

Another interesting case is that of vanishing harmonic oscillator potential. This is achieved using the asymptotic behavior of the ratio of gamma functions for  $\gamma \rightarrow 0$ <sup>39</sup>,

$$\frac{\Gamma\left(\frac{1+|\xi|}{2} - \frac{M\mathcal{E}}{2\gamma}\right)}{\Gamma\left(\frac{1-|\xi|}{2} - \frac{M\mathcal{E}}{2\gamma}\right)} \sim \left(\frac{-M\mathcal{E}}{2\gamma}\right)^{|\xi|}, \quad (81)$$

which holds for  $\mathcal{E} < 0$  and this condition is necessary for the usual AC system has a bound state. Using this limit in Eq. (78) one finds

$$\left(\frac{-M\mathcal{E}}{2\gamma}\right)^{|\xi|} = -\frac{1}{\gamma^{|\xi|} r_0^{2|\xi|}} \left(\frac{\eta + |\xi|}{\eta - |\xi|}\right) \frac{\Gamma(1 + |\xi|)}{\Gamma(1 - |\xi|)}. \quad (82)$$

Then, by solving the Eq. (82) for  $\mathcal{E}$ , we obtain

$$\mathcal{E} = -\frac{2}{Mr_0^2} \left[ \left(\frac{\eta + |\xi|}{\eta - |\xi|}\right) \frac{\Gamma(1 + |\xi|)}{\Gamma(1 - |\xi|)} \right]^{1/|\xi|}. \quad (83)$$

which agree with the result obtained in Eq. (41). Thus, in the limit of vanishing harmonic oscillator, we recover the pure AC problem.

Now we have to remark that this result contains a subtlety that must be interpreted as follows: the presence of the singularity restricts the range of  $\xi$  given by  $\xi \in (-1, 1)$ . If we ignore the singularity and impose that the wave function to be regular at the origin ( $\phi(0) \equiv \dot{\phi}(0) \equiv 0$ ), we achieve at an incomplete answer to spectrum (80), because only the plus sign is presence in the equation and  $\xi$  can have any noninteger value.<sup>44–46</sup>. In this sense, the self-adjoint extension approach prevents us from obtaining a spectrum incompatible with the singular nature of the Hamiltonian at hand when we have (20)<sup>36,47</sup>. We must to take into account that the true boundary condition is that the wave function must be square integrable through all space and it does not matter if it is singular or not at the origin<sup>23,36</sup>.

## V. DETERMINATION OF SELF-ADJOINT EXTENSION PARAMETER

Following the procedure describe in<sup>33</sup>, here we determine the so called self-adjoint extension parameter for the AC problem plus a harmonic oscillator and for the usual AC

system. The approach used in the previous sections give us the energy spectrum in terms of the physics of the problem, but is not appropriate for dealing with scattering problems. Furthermore, it selects the values to parameters  $\varrho$  and  $\vartheta$ . On the other hand, the approach in<sup>48</sup> is suitable to address both bound and scattering scenarios, with the disadvantage of allowing arbitrary self-adjoint extension parameters. By comparing the results of these two approaches for bound states, the self-adjoint extension parameter can be determined in terms of the physics of the problem. Here, all self-adjoint extensions  $\mathcal{H}'_{0,\alpha_\xi}$  of  $\mathcal{H}'_0$  are parametrized by the boundary condition at the origin,

$$\lim_{r \rightarrow 0^+} r^{|\xi|} g(r) = \alpha_\xi \lim_{r \rightarrow 0^+} \frac{1}{r^{|\xi|}} \left\{ g(r) - \left[ \lim_{r' \rightarrow 0^+} r'^{|\xi|} g(r') \right] \frac{1}{r^{|\xi|}} \right\}, \quad (84)$$

where  $\alpha_\xi$  is the self-adjoint extension parameter. In<sup>30</sup> is showed that there is a relation between the self-adjoint extension parameter  $\alpha_\xi$  used here, and the parameters  $\varrho$  and  $\vartheta$  used in the previous sections. The parameters  $\varrho$  and  $\vartheta$ , which are associated with the mapping of deficiency subspaces, who extend the domain of operator to make it self-adjoint, being mathematical parameters. The self-adjoint extension parameter  $\alpha_\xi$  have a physical interpretation, it represents the scattering length<sup>49</sup> of  $\mathcal{H}_{0,\alpha_\xi}$ <sup>30</sup>. For  $\alpha_\xi = 0$  we have the free Hamiltonian (without the  $\delta$  function) with regular wave functions at origin and for  $\alpha_\xi \neq 0$  the boundary condition in (84) permit a  $r^{-|\xi|}$  singularity in the wave functions at origin.

For our intent, it is more convenient to write the solutions for (47) for  $r \neq 0$ , taking into account both cases in (24) simultaneously, as

$$\begin{aligned} \phi_{\mathcal{E}'}(r) = & A_\xi \gamma^{\frac{1+|\xi|}{2}} e^{-\frac{\gamma r^2}{2}} r^{|\xi|} M(d, 1 + |\xi|, \gamma r^2) \\ & + B_\xi \gamma^{\frac{1-|\xi|}{2}} e^{-\frac{\gamma r^2}{2}} r^{-|\xi|} M(d - |\xi|, 1 - |\xi|, \gamma r^2), \end{aligned} \quad (85)$$

where  $A_\xi$ ,  $B_\xi$  are the coefficients of the regular and singular solutions, respectively. By implementing Eq. (85) into the boundary condition (84), we derive the following relation between the coefficients  $A_\xi$  and  $B_\xi$ :

$$\alpha'_\xi A_\xi \gamma^{|\xi|} = B_\xi \left( 1 - \frac{\alpha'_\xi \mathcal{E}'}{4(1 - |\xi|)} \lim_{r \rightarrow 0^+} r^{2-2|\xi|} \right), \quad (86)$$

where  $\alpha'_\xi$  is the self-adjoint extension parameter for the AC plus HO system. In the above equation, the coefficient of  $B_\xi$  diverges as  $\lim_{r \rightarrow 0^+} r^{2-2|\xi|}$ , if  $|\xi| > 1$ . Thus,  $B_\xi$  must be zero for  $|\xi| > 1$ , and the condition for the occurrence of a singular solution is  $|\xi| < 1$ . So, the presence of an irregular solution stems from the fact the operator is not self-adjoint

for  $|\xi| < 1$ , recasting the condition of non-self-adjointness of the previous sections, and this irregular solution is associated with a self-adjoint extension of the operator  $\mathcal{H}'_0$ <sup>50,51</sup>. In other words, the self-adjoint extension essentially consists in including irregular solutions in  $\mathcal{D}(\mathcal{H}'_0)$  to match  $\mathcal{D}(\mathcal{H}_0^\dagger)$ , which allows us to select an appropriate boundary condition for the problem.

In order to Eq. (85) to be a bound state,  $\phi_{\xi, \mathcal{E}'}(r)$  must vanish for large values of  $r$ , i.e., must be normalizable at large  $r$ . By using the asymptotic representation of  $M(a, b, z)$  for  $z \rightarrow \infty$ ,

$$M(a, b, z) \rightarrow \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} + \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^{-a}, \quad (87)$$

the normalizability condition yields the relation

$$B_\xi = -\frac{\Gamma(1+|\xi|)}{\Gamma(1-|\xi|)} \frac{\Gamma\left(\frac{1+|\xi|}{2} - \frac{\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1-|\xi|}{2} - \frac{\mathcal{E}'}{2\gamma}\right)} A_\xi. \quad (88)$$

From Eq. (86), for  $|\xi| < 1$  we have  $B_\xi = \alpha'_\xi \gamma^{|\xi|} A_\xi$  and by using Eq. (88), the bound state energy is implicitly determined by the equation

$$\frac{\Gamma\left(\frac{1+|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)}{\Gamma\left(\frac{1-|\xi|}{2} - \frac{M\mathcal{E}'}{2\gamma}\right)} = -\frac{1}{\alpha'_\xi \gamma^{|\xi|}} \frac{\Gamma(1+|\xi|)}{\Gamma(1-|\xi|)}. \quad (89)$$

By comparing Eq. (89) with Eq. (78), we find

$$\frac{1}{\alpha'_\xi} = \frac{2}{r_0^{2|\xi|}} \left( \frac{\eta + |\xi|}{\eta - |\xi|} \right). \quad (90)$$

We have thus attained a relation between the self-adjoint extension parameter and the physical parameters of the problem. The Eq. (89) coincides with Eq. (53) of Ref.<sup>24</sup> for the AB system with the exact equivalence condition between the vector potential and electric field<sup>4</sup>, i.e.,

$$eA_i = \mu s \epsilon_{ij} E_j. \quad (91)$$

The limiting features of Eq. (90) are interesting. For  $\alpha_\xi \rightarrow 0$  or  $\infty$ , from the poles of gamma function, we have

$$\begin{aligned} \alpha'_\xi = 0, \quad \mathcal{E}' &= (2n + 1 + |\xi|)\omega, \\ \alpha'_\xi = \infty, \quad \mathcal{E}' &= (2n + 1 - |\xi|)\omega. \end{aligned} \quad (92)$$

These bound states energies coincide with those regular and irregular solutions given in Eq. (80) of the previous section. From relation (90) the regular wave function, when  $\alpha'_\xi = 0$  and

the  $\nabla \cdot \mathbf{E}$  is absence, is associated with  $0 < \xi < 1$ , and the irregular wave function, when  $\alpha'_\xi = \infty$ , is associated with  $-1 < \xi < 0$ .

Like before, another interesting limit is that of the vanishing oscillator potential. So, using the limit (59) in (89), we have

$$\frac{1}{2^{|\xi|}} \left( \frac{-M\mathcal{E}}{\gamma} \right)^{|\xi|} = -\frac{1}{\alpha_\xi \gamma^{|\xi|}} \frac{\Gamma(1+|\xi|)}{\Gamma(1-|\xi|)}. \quad (93)$$

with  $\alpha_\xi$  the self-adjoint extension for the usual AC system. From this equation we have the spectrum for the usual AC system in terms of the self-adjoint extension parameter,

$$\mathcal{E} = -\frac{2}{M} \left[ -\frac{1}{\alpha_\xi} \frac{\Gamma(1+|\xi|)}{\Gamma(1-|\xi|)} \right]^{1/|\xi|}. \quad (94)$$

This result also coincides with Eq. (3.13) of Ref.<sup>18</sup> for the AB system, with the exact equivalence cited above. By comparing this equation with (83) we arrive at

$$\frac{1}{\alpha_\xi} = -\frac{2}{r_0^{2|\xi|}} \left( \frac{\eta + |\xi|}{\eta - |\xi|} \right), \quad (95)$$

for the relation of the self-adjoint extension parameter and the physics of the problem for usual AC system. Then, the relation between the self-adjoint extension parameter and the physics of the problem for the usual AC has the same mathematical structure as for the AC plus HO. However, we must observe that the self-adjoint extension parameter is negative for the usual AC, confirming the restriction of negative values of the self-adjoint extension made in<sup>24</sup>, in such way we have an attractive  $\delta$  function. It is a necessary condition to have a bound state in the usual AC system.

It should be mentioned that some relations involving the self-adjoint extension parameter and the  $\delta$  function coupling constant (here represented by  $\eta$ ) were previously obtained by using Green's function in<sup>24</sup> and renormalization technique in<sup>52</sup>, being both, however, deprived from a clear physical interpretation.

## VI. CONCLUSIONS

By modeling the problem by boundary conditions at origin which arises by the self-adjoint extension of the nonrelativistic Hamiltonian, we have presented, for the first time, an expression for the energy spectrum of a spin-1/2 neutral particle with a magnetic moment  $\mu$  moving in a plane subject to an electric field, i.e., for the usual AC system taking into

account the  $\nabla \cdot \mathbf{E}$  term, which features a point interaction between the particle and the charged line. The presence of the  $\nabla \cdot \mathbf{E}$  term, which gives raising of an attractive  $\delta$  function for  $\lambda < 0$ , ensures that we have at least one bound state.

As an application, we also addressed the AC problem plus a two-dimensional harmonic oscillator located at the origin of a polar coordinate system by including the term  $\frac{1}{2}M\omega^2r^2$  in the nonrelativistic Hamiltonian. Two cases were considered: (i) without and (ii) with the inclusion of  $\nabla \cdot \mathbf{E}$  in the nonrelativistic Hamiltonian. Even though we have obtained an equivalent mathematical expression for both cases, has been shown that, in (i)  $\xi$  can assume any value while in (ii) it is in the range  $\xi \in (-1, 1)$ . In the first case, it is reasonable to impose that the wave function vanish at the origin. However, this condition does not give a correct description of the problem in the  $r = 0$  region. Therefore, the energy spectrum obtained in the second case is that physically acceptable.

Finally, at the end we determine the self-adjoint extension parameter for the AC plus HO and for the usual AC problem based on the physics of the problem. Like showed in<sup>33</sup> this self-adjoint extension parameter can be used to study scattering. The scattering in the AC system was studied in<sup>4,21,53</sup> using other methods than self-adjoint extension. Results in this subject using self-adjoint extensions will be reported elsewhere.

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